

A GENERALIZATION OF THE AHLFORS-HEINS THEOREM⁽¹⁾

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1. Notation. Let D be the complex plane cut along the negative real axis, let D' be the upper half-plane and let D'' be the right half-plane. We define

$$D_R = D \cap \{|z| \leq R\}, \quad D'_R = D' \cap \{|z| \leq R\} \quad \text{and} \quad D''_R = D'' \cap \{|z| < R\}.$$

We are going to consider a function u subharmonic in D or D_R . Let $M(r) = \sup_{|z|=r} u(z)$ and $m(r) = \inf_{|z|=r} u(z)$. We also introduce, for $r > 0$,

$$v(r) = \limsup_{z \rightarrow -r+i0} u(z), \quad \tilde{v}(r) = \limsup_{z \rightarrow -r-i0} u(z) \quad \text{and} \quad u(-r) = \max(v(r), \tilde{v}(r)).$$

The symbols $-r+i0$ and $-r-i0$ indicate that z approaches the negative real axis from above and from below, respectively. By v_1 and $\tilde{v}_1$, we mean the corresponding upper limits for the function u_1 . In the whole paper, we write $z = re^{i\theta}$.

In §3, b_1 and b_2 denote positive bounded functions.

2. The main result.

THEOREM. Let λ be a number in the interval $(0, 1)$ and let u ($\neq -\infty$) be a function subharmonic in D that satisfies

$$(2.1) \quad u(-r) - \cos \pi \lambda u(r) \leq 0.$$

Then either $\lim_{r \rightarrow \infty} r^{-\lambda} M(r) = \infty$ or both (A) and (B) hold:

(A) There exists a number α such that

$$(2.2) \quad \lim_{r \rightarrow \infty} r^{-\lambda} u(re^{i\theta}) = \alpha \cos \lambda \theta, \quad |\theta| < \pi,$$

except when θ belongs to a set of logarithmic capacity zero.

(B) Given θ_0 , $0 < \theta_0 < \pi$, there exists an r -set Δ_0 of finite logarithmic length such that (2.2) holds uniformly in $\{z \mid |\theta| \leq \theta_0\}$ when r is restricted to lie outside of Δ_0 .

REMARK. When $1/2 < \lambda < 1$, condition (2.1) is interpreted in the following way at points where $u(-r) = +\infty$.

$$(2.1a) \quad \limsup_{z \rightarrow r} (u(-x+iy) - \cos \pi \lambda u(x+iy)) \leq 0.$$

It follows that we also have

$$(2.1b) \quad \limsup_{z \rightarrow r} (u(x+iy) + u(-x+iy)) \leq (1 + \cos \pi \lambda) u(r).$$

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If we choose $\lambda = 1/2$ and assume that $\liminf_{r \rightarrow \infty} r^{-1/2} M(r)$ is finite, we obtain the Ahlfors-Heins Theorem [1].

For the special case $0 < \lambda < 1/2$, an outline of the proof has been presented under the same title by M. Essén, Bull. Amer. Math. Soc. **75** (1969), 127–131.

The proof of the theorem is long. At the end, we find ourselves in the classical Ahlfors-Heins situation. An important part of the proof is the study of convolution inequalities which will be deduced for u . §§3–5 are devoted to preliminary discussions. In §3, we deduce a generalization of a theorem of Kjellberg on functions subharmonic in the complex plane. In §4, we find the convolution inequalities mentioned above. The lemmas on convolution inequalities which we need are brought together in §5. The proof of the main theorem is given in §§6–9. A reader who wants to follow the main stream of reasoning of this work should start with §§6–9 and use §§3–5 as references.

The present paper is inspired by the papers [6] and [7] of Professor Bo Kjellberg whom the author wants to thank for many stimulating discussions and, in particular, for removing an unnecessary assumption similar to (3.1) from the main theorem. The author also gratefully acknowledges interesting discussions with Mr. John Lewis.

ADDED IN PROOF, MAY 27, 1969. After this work was completed, I received a preprint of a paper "Subharmonic functions in a circle", by Hellsten, Kjellberg and Norstad which a reader interested in the present article would do well to consult. It will appear in the Arkiv för Matematik.

3. Two lemmas on the growth of u at infinity. In the whole paper, we are going to assume that there exists a positive number r_0 such that

(3.1) u is harmonic, bounded and has a negative upper bound in D_{r_0} .

Condition (3.1) is an unessential restriction on u . In fact, any subharmonic function for which (2.1) is true can be modified near the origin in such a way that (2.1) and (3.1) will be true for the new function. The proof is given after the proof of Lemma 3.1 which is related to results in Kjellberg [6].

LEMMA 3.1. *Let $u (\not\equiv -\infty)$ be subharmonic in D and let u satisfy (2.1) and (3.1). Then either $\lim_{r \rightarrow \infty} r^{-\lambda} M(r) = \infty$ or both $\limsup_{r \rightarrow \infty} r^{-\lambda} u(r) < \infty$ and*

$$\limsup_{r \rightarrow \infty} r^{-\lambda} u(\pm ir) < \infty.$$

Proof. We use the technique of Kjellberg [7, p. 8], modified so that we can use it in D_R , where $R > 0$. The subharmonic functions considered here are bounded from above (cf. (2.1b)).

Let w_1 be the harmonic function in D'_R which has boundary values $\frac{1}{2}(u(r) + v(r))$ at $x = \pm r$, $0 \leq r \leq R$, on the real axis and which has boundary values $M(R)$ on the semicircle. The function w_1 is a harmonic majorant of $\frac{1}{2}(u(x + iy) + u(-x + iy))$ in D'_R and we clearly have $u(iy) \leq w_1(iy)$, $0 < y < R$. Let $\tilde{w}_1$ be the corresponding

harmonic function in the lower half-plane. We define w_2 as the function harmonic in D_R'' with boundary values w_1 on the positive imaginary axis, $\tilde{w}_1$ on the negative imaginary axis and $M(R)$ on the semicircle. It is clear that w_1 and $\tilde{w}_1$ are harmonic majorants of u in D_R' and $\bar{D}_R'$, respectively, and that w_2 is a harmonic majorant of u in D_R'' . We can use Poisson's formula for a semicircle (cf. e.g. Boas [2, Theorem 1.2.3]). In D_R'' , we obtain

$$(3.2) \quad u(r) \leq (r/\pi) \int_0^R \left(\frac{1}{t^2 + r^2} - \frac{R^2}{R^4 + r^2 t^2} \right) (w_1(it) + \tilde{w}_1(-it)) dt \\ + (2Rr/\pi) \int_0^\pi \frac{(R^2 - r^2) \sin \phi}{|R^2 e^{2i\phi} + r^2|^2} M(R) d\phi.$$

The last term can be written $(r/R)M(R)b_1(r/R)$, where b_1 is a positive bounded function. In D_R' , we have

$$(3.3) \quad w_1(it) \leq (t/\pi) \int_0^R \left(\frac{1}{t^2 + s^2} - \frac{R^2}{R^4 + t^2 s^2} \right) (u(s) + v(s)) ds + (t/R)M(R)b_1(t/R).$$

By (2.1), $u(s) + v(s) \leq (1 + \cos \pi\lambda)u(s)$. We have similar formulas for w_1 and $u + \tilde{v}$. Eliminating w_1 and $\tilde{w}_1$, we obtain

$$(3.4) \quad u(r) \leq (2r(1 + \cos \pi\lambda)/\pi^2) \int_0^R u(s) \int_0^R t \left(\frac{1}{t^2 + r^2} - \frac{R^2}{R^4 + r^2 t^2} \right) \\ \cdot \left(\frac{1}{s^2 + t^2} - \frac{R^2}{R^4 + t^2 s^2} \right) dt ds + (r/R)(1 + \log(R/r))M(R)b_2(r/R).$$

In the right-hand member, we first replace u by $u^+ = \max(u, 0)$, secondly we the complicated expressions containing R in the inner integral, and thirdly we replace R by ∞ in this integral and divide by r^λ . We obtain

$$(3.5) \quad r^{-\lambda}u(r) \leq \int_0^R s^{-\lambda}u^+(s)K(r, s) ds + R^{-\lambda}M(R)(r/R)^{1-\lambda}(1 + \log(R/r))b_2(r/R)$$

where

$$(3.6) \quad K(r, s) = (2(1 + \cos \pi\lambda)/\pi^2) \frac{(s/r)^\lambda r \log(r/s)}{r^2 - s^2}.$$

The kernel K is the function (18) in Kjellberg [7]. A formal passage to the limit of course gives formula (17) in [7].

If $u^+(r) \equiv 0$, $u(r)$ has a finite upper bound and there is nothing to prove. Suppose that $\sup_{0 < r < R} r^{-\lambda}u(r) = \xi^{-\lambda}u(\xi)$ is positive. By (3.1), $0 < \xi < R$. Using formula (19) in Kjellberg [7], we obtain $\int_0^\infty K(r, s) ds = 1$. It follows from (3.5) that

$$\xi^{-\lambda}u(\xi) \int_R^\infty K(\xi, s) ds \leq \text{Const } R^{-\lambda}M(R)(\xi/R)^{1-\lambda}[\log(R/\xi) + 1].$$

Putting $\xi/R = t$, we obtain if $M(R)$ is positive, that

$$\xi^{-\lambda}u(\xi) \leq \text{Const } R^{-\lambda}M(R) \sup_{0 < t < 1} \left\{ t^{1-\lambda}[\log(1/t) + 1] \left\{ \int_{1/t}^\infty \frac{y^\lambda}{y^2 - 1} \log y dy \right\}^{-1} \right\}.$$

The constants do not depend on R or ξ and hence $\xi^{-\lambda}u(\xi) \leq \text{const } R^{-\lambda}M(R)$. If $\liminf_{R \rightarrow \infty} R^{-\lambda}M(R) < \infty$, it thus follows that $\limsup_{r \rightarrow \infty} r^{-\lambda}u(r) < \infty$. Combining (3.3) and (2.1), we obtain $\limsup_{r \rightarrow \infty} r^{-\lambda}u(\pm ir) < \infty$, and the lemma is proved.

We now prove that (3.1) is an unessential restriction. Important parts of this argument are due to Kjellberg.

In this discussion, let $A(R, \delta) = D \cap \{z \mid ||z| - R| < \delta\}$. We first change the subharmonic function u into a new subharmonic function u_2 which is bounded in a domain of the type $A(R, \delta)$. We observe that if (2.1) is valid for a function u , it is also valid for $u - C$ where C is a positive constant. If we know that u is bounded above in a domain of type $A(R, \delta)$, we can subtract a positive constant and obtain a nonpositive function in this domain which still fulfills (2.1).

We first claim that there exist positive numbers r_1 and δ , $r_1 > 2\delta$, such that the subharmonic function u is bounded above in $A(r_1, 2\delta)$. This is obvious when $0 < \lambda \leq 1/2$. When $1/2 < \lambda < 1$ it follows from (2.1) that there exists a positive number r_1 such that $u(-r_1)$ is not $+\infty$. We know that boundary values of a subharmonic function are upper semicontinuous and hence u is bounded above in a domain $A(r_1, 2\delta)$ for some $\delta > 0$.

We can assume that u is nonpositive in $A(r_1, 2\delta)$. Let g be the harmonic function in $A(r_1, 2\delta)$ which has boundary values u on the two circular arcs and boundary value 0 on the part of the boundary which is on the negative real axis. The function u_2 defined by

$$\begin{aligned} u_2(z) &= g(z), & z \in A(r_1, 2\delta), \\ u_2(z) &= u(z), & z \in D - A(r_1, 2\delta), \end{aligned}$$

is subharmonic in D and is bounded in $A(r_1, \delta)$.

Hence u_2 has an upper bound B and a lower bound b on the closure of

$$\{z \mid r_3 < |z| < r_2\} \cap D.$$

Let $w = \rho e^{i\phi} = \xi + i\eta$. We now map D onto $S = \{w \mid |\phi| < \alpha\pi\}$ using the mapping $w = z^\alpha$ where $\alpha > 0$. A new function U_1 is defined in S by $U_1(w) = u_2(z)$. If the positive number α is small enough, there exists a function V defined by $V(w) = C_1\xi + C_2$, $w \in S$, where C_1 and C_2 are constants, such that $V(w) < b$, $|w| = r_2^\alpha$, $|\phi| \leq \alpha\pi$, and $V(w) > B$, $|w| = r_3^\alpha$, $|\phi| \leq \alpha\pi$. The function U_2 defined in S by

$$\begin{aligned} U_2(w) &= V(w), & |w| < r_3^\alpha, \\ U_2(w) &= \max(U_1(w), V(w)), & r_3^\alpha \leq |w| \leq r_2^\alpha, \\ U_2(w) &= U_1(w), & r_2^\alpha < |w|, \end{aligned}$$

is subharmonic in S . Going back to the z -plane, we obtain a function which is bounded in D_{r_2} and which fulfills (2.1) for $r \geq r_2$. Subtracting a positive constant, we obtain a function subharmonic in D for which (2.1) and (3.1) are true.

The author is indebted to Professor Bo Kjellberg for pointing out the following lemma of well-known type (cf. e.g. Boas [2, Theorem 5.1.2]).

LEMMA 3.2. *Let u be subharmonic in D'' , let $u(it) = \limsup_{z \rightarrow it} u(z)$ and assume that $\liminf_{r \rightarrow \infty} M(r)/r = 0$. We define*

$$\gamma_1 = \limsup_{t \rightarrow +\infty} t^{-\lambda} u(it) \quad \text{and} \quad \gamma_2 = \limsup_{t \rightarrow +\infty} t^{-\lambda} u(-it)$$

and suppose that they are finite. If $\limsup_{r \rightarrow \infty} r^{-\lambda} u(r) = 0$, then $\gamma_1 + \gamma_2 \geq 0$.

Proof. Let $\varepsilon > 0$. Then there exists a sequence $\{r_n\}_1^\infty$, $\lim_{n \rightarrow \infty} r_n = \infty$, and constants A_1 and A_2 such that

$$\begin{aligned} u(r) &> -\varepsilon r^\lambda, & r \in \{r_n\}_1^\infty, \\ u(it) &< A_1 + (\gamma_1 + \varepsilon)t^\lambda, & t > 0, \end{aligned}$$

and

$$u(-it) < A_2 + (\gamma_2 + \varepsilon)t^\lambda, \quad t > 0.$$

Using the Poisson representation formula for a half-plane, we obtain if $r \in \{r_n\}_1^\infty$, that

$$-\varepsilon r^\lambda \leq u(r) \leq (r/\pi) \int_0^\infty \frac{A_1 + A_2 + (\gamma_1 + \gamma_2 + 2\varepsilon)t^\lambda}{t^2 + r^2} dt.$$

Let $r_n \rightarrow \infty$. We obtain

$$-\pi\varepsilon \leq \int_0^\infty \frac{t^\lambda}{t^2 + 1} dt (\gamma_1 + \gamma_2 + 2\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, Lemma 3.2 is proved.

4. Integral inequalities. In this section we deduce several formulas by letting $R \rightarrow \infty$ in formulas like (3.4) and (3.5). We always assume that

$$\liminf_{R \rightarrow \infty} R^{-\lambda} M(R) < \infty$$

and that R tends to infinity through a sequence $\{R_\nu\}_1^\infty$ such that $M(R_\nu) = O(R_\nu^\lambda)$, $\nu \rightarrow \infty$.

Divide (3.4) by r^λ and let $R \rightarrow \infty$. The positive kernel in the integral is less than $K(r, s)$, $r^{-\lambda} u(r)$ is bounded above at infinity and negative near the origin, and hence

$$(4.1) \quad r^{-\lambda} u(r) \leq \int_0^\infty s^{-\lambda} u(s) K(r, s) ds.$$

Since $u \neq -\infty$, the left-hand member is finite for certain values of r and the integral is absolutely convergent.

Suppose that the Poisson integral w of the boundary values of u in the upper half-plane D' exists and is a harmonic majorant of u in D' . The corresponding

majorant in the lower half-plane is called $\tilde{w}$. The nonnegative functions p and $\tilde{p}$ are defined by

$$(4.2) \quad u(z) = w(z) - p(z), \quad \operatorname{Im} z > 0,$$

$$(4.3) \quad u(z) = \tilde{w}(z) - \tilde{p}(z), \quad \operatorname{Im} z < 0.$$

Similarly, let σ be the harmonic function in the right half-plane D'' which has boundary values $u(it)$, $t \in \mathbf{R}$. Under the same assumptions as above, the nonnegative function q is defined by

$$(4.4) \quad u(z) = \sigma(z) - q(z).$$

Repeating the argument leading to (3.4) (with $R = \infty$), we obtain the following relation after division by r^λ .

$$(4.5) \quad \begin{aligned} & 2(1 + \cos \pi \lambda) \left(r^{-\lambda} u(r) - \int_0^\infty s^{-\lambda} u(s) K(r, s) ds \right) \\ &= \int_0^\infty s^{-\lambda} (v(s) + \tilde{v}(s) - 2 \cos \pi \lambda u(s)) K(r, s) ds \\ &\quad - 2(1 + \cos \pi \lambda) \left(r^{-\lambda} q(r) + (r^{1-\lambda}/\pi) \int_0^\infty s^{-\lambda} (p(is) + \tilde{p}(-is)) \frac{s^\lambda}{s^2 + r^2} ds \right). \end{aligned}$$

5. Lemmas on convolution inequalities. From the formulas (4.1) and (4.5), by the change of variables $r = e^x$, $s = e^y$ we can deduce convolution inequalities. In this section, we have brought together some simple results on such inequalities which will be needed in the sequel.

Let K be a nonnegative function in $L^1(-\infty, \infty)$ such that $\int_{-\infty}^\infty K(x) dx = 1$ and $\int_{-\infty}^\infty x K(x) dx = m \neq 0$ where the integrals are assumed to be absolutely convergent. The function N is defined by

$$\begin{aligned} N(x) &= \int_x^\infty K(y) dy, & x > 0, \\ &= -\int_{-\infty}^x K(y) dy, & x < 0. \end{aligned}$$

In Lemma 5.2, we also assume that

$$(5.1) \quad |N(x)| \leq \text{Const } K(x).$$

In the applications in this paper,

$$(5.2) \quad K(x) = (2(1 + \cos \pi \lambda)/\pi^2) \frac{x e^{x(1-\lambda)}}{e^{2x} - 1}, \quad 0 < \lambda < 1,$$

and $m = -\pi \tan(\pi \lambda/2)$ (cf. Essén [4, Formula (3)]). For this function, (5.1) is obviously true with a suitable choice of the constant.

The functions K and $K(r, s)$ of (3.6) are connected by the following formula:

$$(5.3) \quad K(x-y) dy = K(r, s) ds,$$

where $r = e^x$, $s = e^y$.

We study the convolution inequality

$$(5.4) \quad \phi - \phi * K \leq 0.$$

A solution of (5.4) is a locally integrable function ϕ such that $\phi * K$ converges absolutely and the inequality is valid.

For the concept of a slowly decreasing function, we refer to Widder [9, Chapter IV (9b)].

LEMMA 5.1. *Let ϕ be a bounded solution of (5.4). If $\lim_{|x| \rightarrow \infty} \phi(x) = 0$, then $\phi(x) = 0$ a.e.*

Proof. Let $\phi - \phi * K = h$. Arguing as in Essén [4], we obtain

$$(5.5) \quad \int_a^b h(t) dt = N * \phi(b) - N * \phi(a).$$

Let $b \rightarrow +\infty$ and $a \rightarrow -\infty$. Hence $\int_{-\infty}^{\infty} h(t) dt = 0$, and since h is a nonpositive function, $h = 0$ a.e. It follows from (5.5) that $N * \phi = 0$. Since $\hat{N}(t) \neq 0$ (cf. Essén [4]) and $\phi \in L^\infty$, it follows by a standard argument that $\phi(x) = 0$ a.e.

LEMMA 5.2. *Let ϕ be a nonpositive solution of (5.4). If*

(i) $\limsup_{x \rightarrow \infty} \phi(x) = 0$,

(ii) (5.1) is true,

*then $\phi - \phi * K \in L^1(0, \infty)$.*

Proof. Put $b = x$ and $a = 0$ in (5.5). We obtain

$$\phi * N(x) - \phi * N(0) = \int_0^x h(t) dt.$$

By (5.1) and (5.4), we have

$$|\phi * N(x)| \leq \text{const } |\phi * K(x)| \leq \text{const } |\phi(x)|.$$

Hence $\liminf_{x \rightarrow \infty} \left| \int_0^x h(t) dt \right| < \infty$. Since h is nonnegative, $h \in L^1(0, \infty)$, and the lemma is proved.

We define

$$\begin{aligned} \phi_c(x) &= \phi(x), & \phi(x) &\geq -c, \\ &= -c, & \phi(x) &< -c. \end{aligned}$$

LEMMA 5.3. *Let ϕ be a nonpositive solution of (5.4). If (i) $\limsup_{x \rightarrow \infty} \phi(x) = 0$, (ii) there exists a positive constant c such that ϕ_c is slowly decreasing at infinity, then $\lim_{x \rightarrow \infty} \phi(x) = 0$.*

Proof. From (5.4), we deduce that $\phi_c \leq \phi_c * K$. By the same argument as in Essén [4], we conclude that $\lim_{x \rightarrow \infty} \phi_c(x)$ exists. Hence

$$0 = \limsup_{x \rightarrow \infty} \phi(x) \leq \lim_{x \rightarrow \infty} \phi_c(x) \leq 0,$$

$\phi(x) = \phi_c(x)$ when x is large and $\lim_{x \rightarrow \infty} \phi(x) = 0$. Lemma 5.3 is proved.

LEMMA 5.4. *Let ϕ be a solution of*

$$(5.6) \quad \phi - \phi * K \leq -P * K_1 - Q,$$

where (i) $K_1 \in L^1(-\infty, \infty)$ and is nonnegative, (ii) P and Q are nonnegative, locally integrable and $P * K_1$ is defined, (iii) $\phi - \phi * K \in L^1(0, \infty)$. Then P and Q are in $L^1(0, \infty)$.

Proof. Integrating (5.6), we obtain

$$-\infty < - \int_{-\infty}^{\infty} \int_{-y}^{\infty} P(u) du K_1(y) dy - \int_0^{\infty} Q(y) dy,$$

and the lemma is proved.

6. Preliminary study of u . We assume in the remaining part of the paper that $\liminf_{r \rightarrow \infty} r^{-\lambda} M(r) < \infty$. In the other case, $\lim_{r \rightarrow \infty} r^{-\lambda} M(r) = \infty$, and there is nothing to prove.

It follows from Lemma 3.1 that $\alpha = \limsup_{r \rightarrow \infty} r^{-\lambda} u(r) < \infty$. If $\alpha \neq -\infty$, we form $u_1(z) = u(z) - \alpha r^{-\lambda} \cos \lambda \theta$. It is easy to check that the assumptions of the main theorem of this paper are fulfilled by u_1 and hence that the formulas of §4 are valid with u replaced by u_1 .

LEMMA 6.1. α is finite and $u_1(r)$ is nonpositive, $r \geq 0$.

Proof. Inequality (4.1) is valid for u . By the change of variables $r = e^x$, $s = e^y$, we obtain by (5.3) a convolution inequality of the type (5.4) where the kernel K is given by (5.2) and $\phi(x) = e^{-\lambda x} u(e^x)$. Suppose $\alpha = -\infty$. Since $\lim_{x \rightarrow -\infty} \phi(x) = -\infty$, it follows that $\lim_{|x| \rightarrow \infty} (\phi(x) - a)^+ = 0$, where a is an arbitrary real number. From the inequality, we deduce that

$$(\phi - a)^+ \leq (\phi - a)^+ * K.$$

Applying Lemma 5.1, it follows that $(\phi - a)^+ = 0$ a.e. Since $(\phi - a)^+$ is upper semicontinuous, it follows that $(\phi - a)^+ = 0$. This is true for all real numbers a , hence $\phi = -\infty$ and hence also $u(z) \equiv -\infty$, which is impossible. Hence α is finite. Repeating this argument for $(\phi - \alpha)^+$, we obtain that $(\phi - \alpha)^+ = 0$. Hence u_1 is nonpositive and the lemma is proved.

If we can prove that $r^{-\lambda} u_1(re^{i\theta})$ tends to zero when $r \rightarrow \infty$ in the sense of (A) or (B), our theorem will be proved.

7. A lemma on u_1 . We introduce

$$(7.1) \quad \phi_1(x) = e^{-\lambda x} u_1(e^x).$$

Inequality (4.1) is valid for u_1 . By the change of variables $r=e^x$, $s=e^y$, we obtain $\phi_1 - \phi_1 * K \leq 0$, where K is given by (5.2) (also cf. (5.3)). By the remark after (4.1), the integral is absolutely convergent.

The following lemma is an immediate consequence of Lemma 5.2.

LEMMA 7.1. $\phi_1 - \phi_1 * K \in L^1(0, \infty)$.

8. Harmonic majorants in half-planes. We first deal with the more complicated case $1/2 < \lambda < 1$. Consider the subharmonic functions $V_1(z) = u_1(x+iy) + u_1(-x+iy)$ and $V_2(z) = (1 + \cos \pi\lambda)u_1(z) - \cos \pi\lambda V_1(z)$ which have nonpositive boundary values on the real axis (cf. (2.1), (2.1a), and (2.1b)). We have assumed that

$$\liminf_{r \rightarrow \infty} r^{-\lambda} M(r) < \infty.$$

It follows from the Phragmén-Lindelöf theorem (cf. e.g. Heins [5, p. 111]) that V_1 and V_2 are nonpositive in the upper and lower half-planes. Using two representation theorems, one of F. Riesz (cf. Radó [8, §6.19]) and one of Herglotz (cf. Heins [5, Theorem 4.2]), we obtain in D'

$$V_1(z) = -\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha_1(s)}{(x-s)^2 + y^2} - P_1(z) = g_1(z) - P_1(z),$$

and

$$V_2(z) = -\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha_2(s)}{(x-s)^2 + y^2} - P_2(z) = g_2(z) - P_2(z),$$

where α_1 and α_2 are increasing functions and P_1 and P_2 are potentials in D' . It follows from Lemma 3.2 that no y -term appears in the expressions for V_1 and V_2 . Hence

$$(1 + \cos \pi\lambda)u_1 = \cos \pi\lambda g_1 + g_2 - \cos \pi\lambda P_1 - P_2,$$

$(\cos \pi\lambda g_1 + g_2)/(1 + \cos \pi\lambda)$ is the least harmonic majorant of u_1 in D' and it corresponds to the boundary measure

$$d\mu = (-\cos \pi\lambda d\alpha_1 - d\alpha_2)/(1 + \cos \pi\lambda).$$

The purpose of the present section is to prove that the Poisson integral of the boundary values of u_1 in D' is a harmonic majorant of u_1 in D' and that this harmonic majorant has the right kind of behavior at infinity.

We claim that:

I. The singular part μ_s of μ is nonpositive.

II. Let $d\mu = f(s) ds + d\mu_s$ be the canonical decomposition of μ into an absolutely continuous and a singular part. Then $f(s) \leq u_1(s)$ a.e. on $\mathbf{R}$.

It follows immediately from (I) and (II) that the Poisson integral of the boundary values of u_1 in D' is a harmonic majorant of u_1 when $1/2 < \lambda < 1$. When $0 < \lambda \leq 1/2$,

this is a consequence of the Phragmén-Lindelöf theorem (cf. e.g. Heins [5, p. 111]). Similar statements are true in the lower half-plane.

Proof of I. Since u_1 is subharmonic in a region containing the positive real axis, its boundary measure μ is absolutely continuous on this part of the boundary of D' . Hence $d\mu_s = 0$ on $\{z > 0\}$. On the negative real axis, μ_s is the singular part of $-\alpha_1$. This is clear since the second term in the sum defining V_1 corresponds to an absolutely continuous boundary measure on $\{z < 0\}$. Hence (I) is proved.

Proof of II. We apply Theorem 4.3 of Doob [4] to the nonpositive subharmonic functions V_1 and V_2 and the positive harmonic function $z \mapsto r^\lambda \cos \lambda(\theta - \pi/2)$, $\text{Im } z > 0$. We conclude that V_1 and V_2 and hence also u_1 have boundary values a.e. on the real axis in the fine topology. Similarly, the fine limits of the potentials are 0 a.e. on the real axis. By well-known properties of Poisson integrals, the fine limit of u_1 is f a.e. on $\mathbf{R}$. Hence $f(s) \leq \limsup_{z \rightarrow s} u_1(z) \leq u_1(s)$ a.e. on $\mathbf{R}$ and (II) is proved.

We now discuss the behavior of the Poisson integral w of the boundary values of u_1 .

LEMMA 8.1. $\lim_{r \rightarrow \infty} r^{-\lambda} w(re^{i\theta}) = 0$, $0 < \theta < \pi$. The convergence is uniform in each inner sector. A corresponding result is true in the lower half-plane.

Proof. We first deal with the case $1/2 < \lambda < 1$. Define, for functions f such that the integral is absolutely convergent, with $z = x + iy$,

$$P(f)(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2 + y^2} ds.$$

We also introduce the nonnegative function (cf. (2.1))

$$\psi_1(s) = \cos \pi \lambda (u_1(s) + v_1(s)), \quad s \in \mathbf{R},$$

and the nonpositive function

$$\begin{aligned} \psi_2(s) &= u_1(s) - \cos \pi \lambda v_1(s), & s > 0, \\ &= v_1(s) - \cos \pi \lambda u_1(s), & s < 0. \end{aligned}$$

Clearly $w(1 + \cos \pi \lambda) = P(\psi_1) + P(\psi_2)$, and we shall prove that $P(\psi_1)$ and $P(\psi_2)$ both have the right kind of behavior at infinity. We only discuss $P(\psi_2)$. The treatment of $P(\psi_1)$ is similar.

Since ψ_2 is nonpositive, it is sufficient to prove

$$(8.1) \quad \lim_{r \rightarrow \infty} r^{-\lambda} P(\psi_2)(ir) = 0.$$

But $P(\psi_2)(ir) = (1 - \cos \pi \lambda) w_1(ir)$, $r > 0$, where w_1 is the harmonic function in D' with boundary values $\frac{1}{2}(u_1(s) + v_1(s))$, $s \in \mathbf{R}$ (cf. the beginning of the proof of Lemma 3.1 with $R = \infty$). It is sufficient to prove that $\lim_{r \rightarrow \infty} r^{-\lambda} w_1(ir) = 0$.

Also here we apply the technique of Kjellberg [7, p. 8]. It follows from (2.1) that

$$(8.2) \quad w_1(ir) \leq \frac{r(1 + \cos \pi\lambda)}{\pi} \int_0^\infty \frac{u_1(s)}{s^2 + r^2} ds.$$

We have a corresponding inequality in the lower half-plane. In the right half-plane, a similar formula is true:

$$(8.3) \quad u_1(s) \leq \frac{s}{\pi} \int_0^\infty \frac{w_1(it) + \tilde{w}_1(-it)}{t^2 + s^2} ds.$$

Eliminating u_1 from (8.2) and (8.3), we obtain after division by r^λ (cf. (4.1)):

$$(8.4) \quad r^{-\lambda}(w_1(ir) + \tilde{w}_1(-ir)) \leq \int_0^\infty s^{-\lambda}(w_1(is) + \tilde{w}_1(-is))K(r, s) ds,$$

($K(r, s)$ is defined by (3.6)).

Changing variables (cf. §5), we see that the function

$$(8.5) \quad x \mapsto e^{-\lambda x}(w_1(ie^x) + \tilde{w}_1(-ie^x)), \quad x \in \mathbf{R},$$

is a solution of a convolution inequality of the type discussed in §5. Lemma 8.1 will be proved in the case $1/2 < \lambda < 1$ if we can apply Lemma 5.3. We have to check two assumptions. The first one is

$$(8.6) \quad \limsup_{r \rightarrow \infty} r^{-\lambda}(w_1(ir) + \tilde{w}_1(-ir)) = 0.$$

It is clear that $u_1(ir) \leq w_1(ir) \leq 0$. We have a corresponding relation in the lower half-plane. Consider the nonpositive number $\gamma = \limsup_{r \rightarrow \infty} r^{-\lambda}(u_1(ir) + u_1(-ir))$. Since $\limsup_{r \rightarrow \infty} r^{-\lambda}u_1(r) = 0$, it follows from Lemma 3.2 that $\gamma \geq 0$. Hence $\gamma = 0$ and (8.6) is proved.

We also have to check that the function defined by (8.5) has property (ii) of Lemma 5.3. This function can be written

$$\phi(x) = \int_{-\infty}^\infty K_1(x-t)g(t) dt$$

where

$$(8.7) \quad K_1(x) = ((1 + \cos \pi\lambda)/\pi)e^{-\lambda x}(2 \cosh x)^{-1}$$

and g is a nonpositive function. It is easily proved that $|K_1'(x)| \leq \text{Const } K_1(x)$. Thus $|\phi'(x)| \leq \text{Const } |\phi(x)|$, ϕ is uniformly continuous on the set $\{x \mid |\phi(x)| \leq c\}$ and ϕ_c is slowly decreasing at infinity.

Hence we can apply Lemma 5.3 and Lemma 8.1 is proved when $1/2 < \lambda < 1$.

When $0 < \lambda \leq 1/2$, the same proof works, except that no decomposition of w is necessary.

9. The final proof. We now know that u_1 is a function subharmonic in D which has harmonic majorants in the upper and the lower half-plane and which can be written as the sum of the majorant and a subharmonic function as in (4.2) and (4.3). The corresponding formula for the right half-plane is (4.4). By Lemma 8.1, we know the behavior of the majorants and it remains for us to consider the nonpositive subharmonic functions.

By the change of variables $r=e^x$, $s=e^y$ in (4.5), we obtain using (2.1) and (7.1) that

$$(9.1) \quad \phi_1 - \phi_1 * K \leq -Q - P * K_1$$

where $P(x)=e^{-\lambda x}(p(ie^x)+\tilde{p}(-ie^x))$ and $Q(x)=e^{-\lambda x}q(e^x)$. We refer to (5.2) and (8.7) for the definitions of K and K_1 .

We want to prove that the following integrals are convergent, i.e.,

$$(9.2) \quad \int_0^\infty r^{-1-\lambda}(p(ir)+\tilde{p}(-ir)) dr < \infty,$$

$$(9.3) \quad \int_0^\infty r^{-1-\lambda}q(r) dr < \infty.$$

It follows from Lemma 7.1 that $\phi_1 - \phi_1 * K \in L^1(0, \infty)$. Applying Lemma 5.4 we conclude that P and Q are in $L^1(0, \infty)$, e.g. that the integrals (9.2) and (9.3) with 1 instead of 0 as the lower limit are convergent. Since u_1 is harmonic in a neighborhood of the origin (cf. (3.1)), the support of the Riesz mass has a positive distance to the origin and the convergence of the integrals over $(0, 1)$ is not difficult to prove. Hence (9.2) and (9.3) are true.

Consider now the upper half-plane D' and the superharmonic function p . It is an unessential restriction to assume that p has continuous boundary values on the real axis and that these boundary values are 0 (otherwise consider

$$\max(-p(re^{i\theta}), -r \sin \theta).$$

We define a subharmonic function in the following way:

$$h_1(re^{i\theta}) = -p(re^{i\theta}), \quad 0 < \theta \leq \pi/2.$$

When $\pi/2 < \theta < \pi$, define h_1 as the least harmonic majorant of $-p$ in this sector. Clearly h_1 has boundary values zero on the real axis. Let H_1 be the function in $\{z \mid 0 < \theta < 3\pi/2\}$ which we obtain when h_1 is harmonically continued over the negative real axis, e.g. $H_1(z) = -h_1(\bar{z})$ when z is in the third quadrant. $H_1(-ir)$ is positive, and by (9.2),

$$(9.4) \quad \int_0^\infty r^{-1-\lambda}H_1(-ir) dr < \infty.$$

Choose a number α in the interval $(1, 3/2)$ such that $\lambda \cdot \alpha^n = 1$ for some natural number n . Map $\{z \mid 0 < \theta < 3\pi/2\}$ onto $\{w \mid 0 < \phi < 3\pi/(2\alpha)\}$ using the mapping $w = z^{1/\alpha}$. We define $H_2(w) = H_1(z)$. If $z = re^{i\theta}$ and $w = \rho e^{i\phi} = \xi + i\eta$, we see that $r^{-\lambda} H_1(z) = \rho^{-\alpha\lambda} H_2(w)$. The function H_2 is continuous at all points of the negative real axis and it follows from (9.2) and (9.4) that

$$(9.5) \quad \int_0^\infty |H_2(-x)| x^{-1-\alpha\lambda} dx < \infty.$$

We form the Poisson integral

$$P_2(w) = \frac{1}{\pi} \int_{-\infty}^0 \frac{H_2(t)\eta}{(\xi-t)^2 + \eta^2} dt.$$

P_2 is positive in D' and we deduce from (9.5) that

$$\int_0^\infty |P_2(it)| t^{-1-\alpha\lambda} dt < \infty.$$

Finally, we define $h_2 = H_2 - P_2$. The function h_2 is subharmonic in D' , it has boundary values 0 and

$$\int_0^\infty t^{-1-\alpha\lambda} |h_2(it)| dt < \infty.$$

Hence h_2 has exactly the same properties in D' as p_1 except that the exponent λ has been replaced by $\lambda\alpha$. Repeating this construction n times, we obtain a function h_{n+1} subharmonic in D' such that

$$(9.6) \quad r^{-\lambda} p(re^{i\theta}) = \rho^{-1} h_{n+1}(\rho e^{i\phi}) + \varepsilon(\rho e^{i\phi}), \quad 0 < \theta \leq \pi/2, \quad 0 < \phi \leq \pi\lambda/2,$$

where $\theta = \phi\alpha^n$, $r = \rho\alpha^n$ and $\lim_{\rho \rightarrow \infty} \varepsilon(\rho e^{i\phi}) = 0$, uniformly in each inner sector of the half-plane. It follows from the construction that

$$(9.7) \quad \int_0^\infty \rho^{-2} |h_{n+1}(\rho e^{i\phi})| d\rho < \infty, \quad 0 < \phi < \pi.$$

The function h_{n+1} has boundary values zero, it is nonpositive and hence the conclusions of the Ahlfors-Heins theorem [1] are true for h_{n+1} . Due to (9.7), we thus have $\lim_{\rho \rightarrow \infty} \rho^{-1} h_{n+1}(\rho e^{i\phi}) = 0$, $0 < \phi < \pi$, in the sense of (A) and (B). Since the properties of the exceptional sets which we discuss here are invariant under the mappings which have been used, it follows from (9.6) that $\lim_{r \rightarrow \infty} r^{-\lambda} u(re^{i\theta}) = 0$, in the sense of (A) and (B) when $0 < \theta \leq \pi/2$.

Continuing P over the positive real axis, we get the corresponding result in the second quadrant. Obviously the same proof applies in the lower half-plane.

We have proved our theorem but for one detail: we have not proved (B) as far as the positive real axis is concerned. This omission is obviously purely technical. Since we have (9.3), the function q in the right half-plane has the same properties as the functions p and $\tilde{p}$ and the preceding argument is valid also for q . By Lemma

8.1 and (9.2), the behavior of the boundary values on the imaginary axis of the least harmonic majorant in D'' is well known, and the proof of the theorem is complete.

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